

FIRST SEMESTER EXAMINATION, 2008-09

MATHEMATICS-I

Time : 3 Hours

Total Marks : 100

- Note : (i) Attempt all questions.
(ii) All questions carry equal marks.
(iii) Be precise in your answer.
(iv) No second answer book will be provided.

SECTION—A

Q. 1. All parts of this question are compulsory.
 $2 \times 10 = 20$

1. (a) For which value of 'b' the rank of the matrix

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix} \text{ is } 2, b =$$

Ans. Justification :

\therefore Rank of A = 2 < order of matrix

$$\Rightarrow \begin{vmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{vmatrix} = 0$$

$$\Rightarrow 1(30 - 26) - 5(-2b) + 4(-3b) = 0$$

$$\Rightarrow 4 + 10b - 12b = 0$$

$$\Rightarrow b = 2$$

1. (b) Determine the constants a and b such that the curl of vector $\vec{A} = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}$ is zero, $a =$, $b =$

Ans. Justification :

$$\therefore \nabla \times \vec{A} = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + 3yz & x^2 + axz - 4z^2 & -3xy - byz \end{vmatrix} = 0$$

$$\Rightarrow \hat{i}[-3x - b^2 - 9x + 8z] - \hat{j}[-3y + 3y] + \hat{k}[2x + 9z - 2x - 3z] = 0$$

$$\Rightarrow \hat{i}[(-3 - a)x + (B - b)z] + \hat{k}(a - 3)z = 0$$

$$\Rightarrow a = -3, b = 8$$

1. (c) The n^{th} derivative (y_n) of the function $y = x^2 \sin x$ at $x = 0$ is

Ans. Justification: $y = x^2 \sin x$

$$\Rightarrow y_n = x^2 \frac{d^n}{dx^n}(\sin x) + {}^n C_1 (2x) \frac{d^{n-1}}{dx^{n-1}}(\sin x)$$

$$+ {}^n C_2 (2) \frac{d^{n-2}}{dx^{n-2}}(\sin x)$$

$$y_n(x) = x^2 \sin\left(x + \frac{nx}{2}\right) + n(2x) \sin\left[x + (n-1)\frac{\pi}{2}\right] + n(n-1) \sin\left[x + (n-2)\frac{\pi}{2}\right]$$

$$\begin{aligned}\Rightarrow y_n &= n(n-1)\sin(n-2)\frac{\pi}{2} \\ &= -n(n-1)\sin\frac{n\pi}{2} \\ &= (n-n^2)\sin\frac{n\pi}{2}\end{aligned}$$

Q. 1. (d) With usual notations, match the items on right hand side with those on left hand side for properties of maximum and minimum.

- | | |
|------------------------|--------------------------------|
| (i) Maxim. (s) | (p) $rt - s^2 = 0$ |
| (ii) Minm. (r) | (q) $rt - s^2 < 0$ |
| (iii) Saddle point (q) | (r) $rt - s^2 > 0$ and $r > 0$ |
| (iv) Failure case (p) | (s) $rt - s^2 > 0$ and $r < 0$ |
- Ans. (i) Max^m (s) (p) $rt - s^2 > 0$ and $r < 0$
- (ii) Min^m (r) (r) $rt - s^2 > 0$ and $r > 0$
- (iii) Saddle point (q) $rt - s^2 < 0$
- (iv) Failure case (p) $rt - s^2 = 0$

1. (e) Match the items on the right hand side with those on left hand side for the following special functions: (Full marks is awarded if all matchings are correct).

- | | |
|---|--|
| (i) $\beta(p, q)$ | (p) $\sqrt{(1/2)}$ |
| (ii) $\frac{\sqrt{p} \sqrt{q}}{\sqrt{p+q}}$ | (q) $\int_0^\infty \frac{y^{p-1}}{(1+y)^{(p+q)}} dy$ |
| (iii) $\sqrt{\pi}$ | (r) $\beta(p, q)$ |
| (iv) $\frac{\pi}{\sin p\pi}$ | (s) $\sqrt{p} \sqrt{1-p}$ |

Ans. (i) $\beta(p, q)$ (p) $\beta(p, q)$

(ii) $\frac{\sqrt{p} \sqrt{q}}{\sqrt{p+q}}$ (q) $\int_0^\infty \frac{y^{p-1}}{(1+y)^{(p+q)}} dy$

(iii) $\sqrt{\pi}$ (r) $\beta(p, q)$

(iv) $\frac{\pi}{\sin p\pi}$ (s) $\sqrt{p} \sqrt{1-p}$

Indicate True or False for the following statements:

Q. 1. (f) (i) If $|A| = 0$, then at least one eigen value is zero. (True/ False)

(ii) A^{-1} exists iff 0 is an eigen value of A. (True/ False)

(iii) If $|A| \neq 0$, then A is known as singular matrix. (True/ False)

(iv) Two vectors X and Y is said to be orthogonal Y, $X^T Y = Y^T X \neq 0$. (True/ False)

Ans. (i) True
(ii) False
(iii) False
(iv) False

Q. 1. (g) (i) The curve $y^2 = 4ax$ is symmetric about x-axis. (True/ False)

(ii) The curve $x^3 + y^3 = 3axy$ is symmetric about the line $y = -x$. (True/ False)

(iii) The curve $x^2 + y^2 = a^2$ is symmetric about both the axis x and y. (True/ False)

(iv) The curve $x^3 - y^3 = 3axy$ is symmetric about the line $y = x$. (True/ False)

Ans. (i) True
(ii) False
(iii) True
(iv) False

Pick the correct answer of the choices given below:

Q. 1. (h) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is position vector, then value of $\nabla(\log r)$ is

(i) $\frac{\vec{r}}{r}$ (ii) $\frac{\vec{r}}{r^2}$

(iii) $-\frac{\vec{r}}{r^3}$ (iv) none of the above

Ans. (ii) $\frac{\vec{r}}{r^2}$

Justification :

$$\nabla(\log r) = \Sigma \hat{i} \frac{\partial}{\partial x} (\log r)$$

$$= \Sigma \hat{i} \frac{\partial}{\partial r} (\log r) \frac{\partial r}{\partial x}$$

$$\begin{aligned}
 &= \Sigma \hat{i} \frac{1}{r} \cdot \frac{x}{r} \left[\because \frac{\partial r}{\partial x} = \frac{x}{r} \right] \\
 &= \frac{15}{r^2} x \hat{i} \\
 &= \frac{\bar{r}}{r^2}
 \end{aligned}$$

1. (i) The Jacobian $\frac{\partial(uv)}{\partial(xy)}$ for the function

$$u = e^x \sin y, \quad v = (x + \log \sin y) \text{ is}$$

- (i) 1
- (ii) $\sin x \sin y - xy \cos x \cos y$
- (iii) 0
- (iv) $\frac{e^x}{x}$

Ans. (iii) 0

Justification: $\partial u = e^x \sin y, y = x + \log \sin y$

$$\begin{aligned}
 \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \\
 &= \begin{vmatrix} e^x \sin y & e^x \cos y \\ 1 & \frac{1}{\sin y} \cos y \end{vmatrix} = 0
 \end{aligned}$$

1. (j) The volume of the solid under the surface $az = x^2 + y^2$ and whose base R is the circle $x^2 + y^2 = a^2$ is given as

- (i) $\pi | 2a$
- (ii) $\pi a^3 | 2$
- (iii) $\frac{4}{3} \pi a^3$
- (iv) None of the above

Ans. (ii) $\frac{\pi a^3}{2}$

Justification : Required Volume = $\iiint dx dy dz$

$$\begin{aligned}
 &= \iiint_R \frac{x^2 + y^2}{a} dx dy dz \\
 &= \iint_R \frac{x^2 + y^2}{a} dx dy \\
 &= \int_{r=0}^a \int_{\theta=0}^{2\pi} \frac{r^2 r dr d\theta}{a} \left[\begin{array}{l} \text{Put } x = r \cos \theta \\ y = r \sin \theta \end{array} \right] \\
 &= \frac{\pi a^3}{2}
 \end{aligned}$$

SECTION-B

Q. 2. Attempt any three parts of the following:

$$10 \times 3 = 30$$

2. (a) If $y = (\sin^{-1} x)^2$, prove that $y_n(0) = 0$ for b odd and $y_n(0) = 2, 2^2, 4^2, 6^2, \dots, (n-2)^2, n \neq 2$ for n is even.

Ans. $\therefore y = (\sin^{-1} x)^2 \dots(1)$

$$\Rightarrow y_1 = 2(\sin x) \cdot \frac{d}{dx} (\sin^{-1} x)$$

$$\Rightarrow y_1 = 2(\sin^{-1} x) \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2} y_1 = 2 \sin^{-1} x$$

$$\Rightarrow (1-x^2) y_1^2 = 3(\sin^{-1} x)^2$$

$$\Rightarrow (1-x^2) y_1^2 = 4y \dots(2) \left[\because y = (\sin^{-1} x)^2 \right]$$

Differentiating (2), we get

$$\frac{d}{dx} [(1-x^2) y_1^2] = \frac{d}{dx} (4y)$$

$$\Rightarrow -2xy_1^2 + (1-x^2) \frac{d}{dx} (y_1^2) = 4y_1$$

$$\Rightarrow -2xy_1^2 + (1-x^2) 2y_1 y_2 = 4y_1$$

$$(1-x^2)y_2 - xy - 2 = 0 \quad \dots(3)$$

Differentiating (3) n times by

$$[(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n]$$

$$-[xy_{n+1} + {}^nC_1(1)y_n] - 0 = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}$$

$$-xy_{n+1} - ny_n = 0$$

$$\left[\therefore {}^nC_1 = n, {}^nC_2 = \frac{n(n-1)}{2} \right]$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \quad \dots(4)$$

$$\text{From (1), } y(0) = [\sin^{-1}(0)]^2 = 0$$

$$\text{From (2), } y_1(0) = 0$$

$$\text{Putting } x = 0 \text{ in (3), } y_2(0) = 2$$

$$\text{Putting } x = 0 \text{ in (4),}$$

$$y_{n+2}(0) = n^2y_n(0) = 0$$

$$\Rightarrow y_{n+2}(0) = n^2y_n(0) \quad \dots(5)$$

$$\text{Putting } n = 1 \text{ in (5), } y_3(1) = 1^2y_1(0) = 0$$

$$\text{Putting } n = 3 \text{ in (5); } y_5(0) = 3^2y_3(0) = 0$$

$$\text{Putting } n = 5 \text{ in (5), } y_7(0) = 5^2y_5(0) = 0$$

$$\Rightarrow y_n(0) = 0 \text{ if } n \text{ is odd.}$$

$$\text{Again, putting } n = 2 \text{ in (5), } y_4(0) = 2^2y_2(0) = 2^2$$

$$\text{Putting } n = 4 \text{ in (5), } y_6(0) = 4^2y_4(0) = 2.2^2.4^2$$

$$\text{Putting } n = 6 \text{ in (5), } y_8(0) = 6^2y_6(0) = 2.2^2.4^2.6^2$$

$$\vdots$$

$$\text{Putting } n = n-2 \text{ in (5),}$$

$$y_n(0) = \dots = 2.2^2.4^2.6^2 - (n-2)^2 \text{ if } n \text{ is even}$$

Thus,

$$y_n(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2.2^2.4^2.6^2 \dots (n-2)^2 & \text{if } n \text{ is even} \end{cases}$$

Q. 2. (b) Find the dimension of rectangular box of maximum capacity whose surface area is

given when (a) box is open at the top (b) box is closed.

Ans. Let x , y and z be the length, breadth and height of the box. Let S & Y be surface area & volume of box respectively.

(a) When box is open at the top.

Then

$$S = 2(xy + yz + zx) - xy$$

$$S = xy + 2(x + y)z \quad \dots(1)$$

$$\text{and } Y = xyz \quad \dots(2)$$

By Lagrange's method,

$$\frac{\partial Y}{\partial x} + \lambda \frac{\partial S}{\partial x} = 0, \frac{\partial Y}{\partial y} + \lambda \frac{\partial S}{\partial y} = 0, \frac{\partial Y}{\partial z} + \lambda \frac{\partial S}{\partial z} = 0$$

$$\Rightarrow yz + \lambda(y + 2z) = 0 \quad \dots(3)$$

$$xz + \lambda(x + 2z) = 0 \quad \dots(4)$$

$$xy + 2\lambda(x + y) = 0 \quad \dots(5)$$

Eqⁿ(3) - Eqⁿ(4) gives

$$(y - x)z + \lambda(y - x) = 0$$

$$\Rightarrow y = x.$$

$$x(2z - y) + 2x(2z - y) = 0$$

$$\Rightarrow y = 2z$$

\Rightarrow For surface area (S) given, the dimensions for maximum capacity are $x = y = 2z$.

Now

$$S = xy + 2(x + y)z \quad [\text{From (1)}]$$

$$= (2z)(2z) + 2(2z + 2z)z$$

$$= 4z^2 + 8z^2$$

$$= 12z^2$$

$$\Rightarrow 8z = \frac{\sqrt{5}}{\sqrt{12}}$$

$$\Rightarrow z = \frac{\sqrt{5}}{2\sqrt{3}}$$

$$\therefore \text{Length } x = 2z = \frac{2\sqrt{5}}{2\sqrt{3}} = \frac{\sqrt{6}}{\sqrt{3}}$$

$$\text{Breadth } y = 2z = \frac{\sqrt{5}}{\sqrt{3}}$$

$$\text{Height } z = \frac{\sqrt{5}}{2\sqrt{3}}$$

(b) When box is closed,
then

$$S = 2(xy + yz + zx) \quad \dots(1)$$

$$\text{and } Y = xyz \quad \dots(2)$$

\therefore By Lagrange's method,

$$\frac{\partial S}{\partial x} + \lambda \frac{\partial V}{\partial x} = 0$$

$$\frac{\partial S}{\partial y} + \lambda \frac{\partial V}{\partial y} = 0$$

$$\frac{\partial S}{\partial z} + \lambda \frac{\partial V}{\partial z} = 0$$

$$\Rightarrow 2(y+z) + \lambda yz = 0 \quad \dots(3)$$

$$2(x+z) + \lambda xz = 0 \quad \dots(4)$$

$$2(x+y) + \lambda xy = 0 \quad \dots(5)$$

Eqⁿ(3) - Eqⁿ(4) gives

$$2(y-x) + \lambda z(y-x) = 0$$

$$\Rightarrow y = x.$$

Eqⁿ(4) - Eqⁿ(5) gives

$$2(z-y) + \lambda x(z-y) = 0$$

$$\Rightarrow y = z$$

Thus, $x = y = z.$

$$\therefore S = 2(xy + yz + zx)$$

$$\Rightarrow S = 2(z.z + z.z + z.z) \quad [\because x = y = z]$$

$$\Rightarrow S = 6z^2$$

$$\Rightarrow z = \sqrt{\frac{5}{6}}$$

$$\therefore \text{height of box } x = \sqrt{\frac{5}{6}}$$

$$\text{Breadth of box } y = \sqrt{\frac{5}{6}}$$

$$\text{Height of box } z = \sqrt{\frac{5}{6}}$$

Q. 2. (c) Find a matrix P which diagonalizes the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, verify $P^{-1}AP = D$ where D is the diagonal matrix.

$$\text{Ans. Given matrix } A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Characteristic eqn of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(3-\lambda) - 2 = 0$$

$$\Rightarrow 12 - 4\lambda - 3\lambda + \lambda^2 - 2 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 2\lambda + 10 = 0$$

$$\Rightarrow \lambda(\lambda - 5) - 2(\lambda - 5) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = 2, \lambda = 5, \text{ which are eigen values of}$$

matrix A.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigen vector of A for eigen value

$\lambda.$

$$\Rightarrow (A - \lambda I) X = 0$$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dots(1)$$

For $\lambda = 2$, (1) gives

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_2 = 0$$

$$\text{or } 2x_1 + x_2 = 0$$

$$\text{Let } x_1 = k \Rightarrow x_2 = -2k$$

$$\therefore \text{Eigen vector for } \lambda = 2 \text{ is } \begin{bmatrix} k \\ -2k \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Again, for $\lambda = 5$ (1) gives

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -1x_1 + x_2 = 0$$

$$\text{and } 2x_1 - 2x_2 = 0$$

$$\text{or } x_1 - x_2 = 0$$

$$19 \text{ Let } x_1 = k \Rightarrow x_2 = k$$

$$\therefore \text{Eigen vector for } \lambda = 5 \text{ is } \begin{bmatrix} k \\ k \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \text{Modal matrix } P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, \text{ which will}$$

diagonalize A.

Verification :

We shall calculate $P^{-1}AP$ and it should come as diagonal matrix (D).

$$\text{Now } P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\Rightarrow \text{adj } P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$|P| = (1)(1) + (2)(1) = 3$$

$$\therefore P^{-1} = \frac{\text{adj } P}{|P|} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4-2 & 4+1 \\ 2-6 & 2+3 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -4 & 5 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2+4 & 5-5 \\ 4-4 & 10+5 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Hence verified that $P^{-1}AP$ is a diagonal matrix.

2. (d) Find the area and the mass contained in the first quadrant enclosed by the curve

$$\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta = 1 \text{ where } \alpha > 0, \beta > 0 \text{ given that}$$

density at any point $P(xy)$ is $k\sqrt{xy}$.

$$\text{Ans. Given curve is } \left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta = 1.$$

$$\text{and Density } p(x, y) = k\sqrt{xy}.$$

$$\text{Required area } A = \iint_R dx dy$$

where R is the region given by

$$x \geq 0, y \geq 0, \left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta \leq 1.$$

$$\text{Put } \left(\frac{x}{a}\right)^\alpha = X$$

$$\left(\frac{y}{b}\right)^\beta = Y$$

$$\Rightarrow \frac{x}{a} = X^{\frac{1}{\alpha}}$$

$$\frac{y}{b} = Y^{\frac{1}{\beta}}$$

$$\Rightarrow x = aX^{\frac{1}{\alpha}}$$

$$y = bY^{\frac{1}{\beta}}$$

$$\Rightarrow dx = \frac{a}{\alpha} X^{\frac{1}{\alpha}-1} dX \quad dy = \frac{b}{\beta} Y^{\frac{1}{\beta}-1} dY$$

$$\therefore \text{Required area } A = \iint_{R^1} \frac{a}{\alpha} X^{\frac{1}{\alpha}-1} \frac{b}{\beta} Y^{\frac{1}{\beta}-1} dx dy$$

where R^1 is the region $x > 0, y > 0, x + y \leq 1$.

$$\Rightarrow \text{Required area} = \frac{ab}{\alpha\beta} \iint_{R^1} X^{\frac{1}{\alpha}-1} Y^{\frac{1}{\beta}-1} dx dy$$

$$= \frac{ab}{\alpha\beta} \frac{\sqrt{\frac{1}{\alpha}} \sqrt{\frac{1}{\beta}}}{\sqrt{\frac{1}{\alpha} + \frac{1}{\beta} + 1}}$$

[By Dirichlet's integral]

2nd Part :

$$= \iint_R k \sqrt{xy} dx dy$$

where R is the region

$$x \geq 0, y \geq 0, \left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta \leq 1.$$

$$\text{Put } \left(\frac{x}{a}\right)^\alpha = X, \quad \left(\frac{y}{b}\right)^\beta = Y$$

$$\Rightarrow x = aX^{\frac{1}{\alpha}} \quad y = bY^{\frac{1}{\beta}}$$

$$\Rightarrow dx = \frac{a}{\alpha} X^{\frac{1}{\alpha}-1} dX, \quad dy = \frac{b}{\beta} Y^{\frac{1}{\beta}-1} dY$$

\therefore Required mass

$$M = \iint aX^{\frac{1}{\alpha}} bY^{\frac{1}{\beta}} \frac{a}{\alpha} X^{\frac{1}{\alpha}-1} \frac{b}{\beta} Y^{\frac{1}{\beta}-1} dx dy$$

where R^1 is the region $x > 0, y > 0, x + y \leq 1$.

\Rightarrow Required mass

$$M = k \iint_{R^1} \frac{a^{\frac{1}{\alpha}+1} b^{\frac{1}{\beta}+1}}{\alpha\beta} X^{\frac{1}{2\alpha}+\frac{1}{\alpha}-1} Y^{\frac{1}{2\beta}+\frac{1}{\beta}-1} dx dy$$

$$= \frac{ka^{\frac{3}{2}+1} b^{\frac{3}{2}+1}}{\alpha\beta} \iint_{R^1} X^{\frac{3}{2\alpha}-1} Y^{\frac{3}{2\beta}-1} dx dy$$

$$= \frac{ka^{\frac{3}{2}+1} b^{\frac{3}{2}+1}}{\alpha\beta} \frac{\sqrt{\frac{3}{2\alpha}} \sqrt{\frac{3}{2\beta}}}{\sqrt{\frac{3}{2\alpha} + \frac{3}{2\beta} + 1}} \quad [\text{By Dirichlet's theorem}]$$

2. (e) Using the divergence theorem, evaluate the surface integral

$$\iint_S (yz dy dz + zx dz dx + xy dy dx)$$

where $S: x^2 + y^2 + z^2 = 4$

Ans. Given surface integral

$$I = \iint_S (yz dy dz + zx dz dx + xy dy dx)$$

$$= \iint_S \vec{F} \cdot \hat{n} ds$$

where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

By divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} dy$$

Now,

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) \\ &= 0 \end{aligned}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} ds = 0$$

$$\Rightarrow \iint_S (yz dy dz + zx dz dx + xy dy dx) = 0.$$

SECTION-C

Q. 3. Attempt any two parts from each question. All questions are compulsory.

3. (a) Trace the curve $r^2 = a^2 \cos 2\theta$

Ans. Please see Q.3 (d) of 2000-2001.

3. (b) If $u = \log \left(\frac{x^2 + y^2}{x + y} \right)$, prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1 \quad (\text{Unit-1})$$

Ans. Here $u = \log \frac{x^2 + y^2}{x + y}$

$$\Rightarrow e^u = \frac{x^2 + y^2}{x + y}$$

Let $V(x, y) = \frac{x^2 + y^2}{x + y}$

$$\begin{aligned} \Rightarrow V(tx, ty) &= \frac{t^2(x^2 + y^2)}{t(x + y)} \\ &= t \left(\frac{x^2 + y^2}{x + y} \right) \\ &= t V(x, y) \end{aligned}$$

$\therefore V(x, y)$ is a function of degree $n = 1$.

\therefore By Euler's theorem,

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = \lambda V$$

$$\Rightarrow x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = V$$

$$\Rightarrow x \frac{\partial}{\partial x} (e^u) + y \frac{\partial}{\partial y} (e^u) = e^u$$

$$\left[\therefore Y = e^u = \frac{x^2 + y^2}{x + y} \right]$$

$$\Rightarrow x \frac{\partial}{\partial u} (e^u) \frac{\partial u}{\partial x} + y \frac{\partial}{\partial u} (e^u) \frac{\partial u}{\partial y} = e^u$$

$$\Rightarrow x e^u \frac{\partial u}{\partial x} + y e^u \frac{\partial u}{\partial y} = e^4$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1$$

Hence proved.

3. (c) If $V = f(2x - 3y, 3y - 4z, 4z - 2x)$, compute the value of $6V_x + 4V_y + 3V_z$.

Ans. Here $V = f(2x - 3y, 3y - 4z, 4z - 2x)$

Let $2x - 3y = r, 3y - 4z = s, 4z - 2x = t$

$$\therefore V = f(r, s, t)$$

Now,

$$\begin{aligned} V_x &= \frac{\partial V}{\partial x} \\ &= \frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial V}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial V}{\partial t} \cdot \frac{\partial t}{\partial x} \\ &= \frac{\partial V}{\partial r} \cdot (2) + \frac{\partial V}{\partial s} \cdot (0) + \frac{\partial V}{\partial t} \cdot (-2) \end{aligned}$$

$$\Rightarrow 6 \times x = 12 \frac{\partial V}{\partial r} - 12 \frac{\partial V}{\partial t} \quad \dots(1)$$

Also,

$$\begin{aligned} V_y &= \frac{\partial V}{\partial y} \\ &= \frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial V}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial V}{\partial t} \cdot \frac{\partial t}{\partial y} \\ &= \frac{\partial V}{\partial r} \cdot (-3) + \frac{\partial V}{\partial s} \cdot (3) + \frac{\partial V}{\partial t} \cdot (0) \end{aligned}$$

$$\Rightarrow 4V_y = -12 \frac{\partial V}{\partial r} + 12 \frac{\partial V}{\partial s} \quad \dots(2)$$

Again,

$$\begin{aligned} V_z &= \frac{\partial V}{\partial z} \\ &= \frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial V}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial V}{\partial t} \cdot \frac{\partial t}{\partial z} \end{aligned}$$

$$= \frac{\partial V}{\partial r} (0) + \frac{\partial V}{\partial s} (-4) + \frac{\partial V}{\partial t} (4)$$

$$\Rightarrow 3y_z = -12 \frac{\partial V}{\partial s} + 12 \frac{\partial V}{\partial t} \quad \dots(3)$$

Adding (1), (2) and (3), we get

$$6V_x + 4V_y + 3V_z = 0.$$

Q. 4. (a) The temperature 'T' at any point (xyz) in space is $T(xyz) = Kxyz^2$ where K is constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$

Ans. Here temperature $T = kxyz^2$.

$$\text{Given sphere is } x^2 + y^2 + z^2 = a^2 \quad \dots(1)$$

$$\Rightarrow x^2 + y^2 + z^2 - a^2 = 0.$$

$$\text{Let } \phi = x^2 + y^2 + z^2 - a^2$$

We want to find the highest value of T .

\therefore By Lagrange's method,

$$\frac{\partial T}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial T}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial T}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$$\Rightarrow kyz^2 + 2\lambda x = 0 \quad \dots(2)$$

$$kxz^2 + 2\lambda y = 0 \quad \dots(3)$$

$$2kxy^2 + 2\lambda z = 0 \quad \dots(4)$$

$$\text{From (2), } -2\lambda = \frac{kyz^2}{x} \quad \dots(5)$$

$$\text{From (3), } -2\lambda = \frac{kxz^2}{y} \quad \dots(6)$$

$$\text{From (4), } -2\lambda = \frac{2kxy^2}{z} \quad \dots(7)$$

From (5) and (6),

$$\frac{kyz^2}{x} = \frac{kxz^2}{y}$$

$$y^2 = x^2$$

Also from (6) and (7)

$$\Rightarrow z^2 = 2y^2$$

$$\Rightarrow z^2 = 2x^2 \quad [\because y^2 = x^2]$$

Now, from,

$$x^2 + y^2 + z^2 = a^2$$

$$\Rightarrow x^2 + x^2 + 2x^2 = a^2$$

$$\Rightarrow x^2 = \frac{a^2}{4}$$

$$\Rightarrow x = \pm \frac{a}{2}$$

$$y^2 = x^2$$

$$\Rightarrow y = \pm \frac{a}{2}$$

$$\text{Also, } z^2 = 2x^2$$

$$\Rightarrow z^2 = \frac{a^2}{2}$$

$$\Rightarrow z = \pm \frac{a}{\sqrt{2}}$$

$$T = kxyz^2$$

$$\Rightarrow T_{\max} = k \left(\pm \frac{a}{2} \right) \left(\pm \frac{a}{2} \right) \left(\pm \frac{a}{\sqrt{2}} \right)^2$$

$$\Rightarrow T_{\max} = \frac{ka^4}{4}$$

Q. 4. (b) Verify the chain rule for Jacobians if

$$x = u, y = u \tan v, z = w$$

Ans. We shall verify

$$\Rightarrow \frac{\partial(x, y, z)}{\partial(u, v, w)} \times \frac{\partial(u, v, w)}{\partial(x, y, z)} = 1$$

Here $x = u, y = u \tan v, z = w$

$$\Rightarrow \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \tan u & 4 \sec^2 u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 4 \sec^2 u$$

To calculate $\frac{\partial(u, v, w)}{\partial(x, y, z)}$, we shall express u, v and w in terms of x, y and z .

$$\therefore x = u \Rightarrow u = x$$

Also, $y = u \tan v$

$$\Rightarrow \tan v = \frac{y}{u}$$

$$= \frac{y}{x}$$

$$\Rightarrow v = \tan^{-1} \frac{y}{x}$$

Again,

$$z = w$$

$$\Rightarrow w = z$$

Thus, we have obtained

$$u = x, v = \tan^{-1} \frac{y}{x}, w = z.$$

Now, $\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \left(1 + \frac{y^2}{x^2}\right) \left(\frac{-y}{x^2}\right) & \frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{1}{x}\right) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{x}{x^2 + y^2}$$

$$= \frac{u}{u^2 + u^2 \tan^2 v}$$

$$= \frac{u}{u^2 (1 + \tan^2 v)}$$

$$= \frac{1}{4 \sec^2 v}$$

$$\Rightarrow \frac{\partial(x, y, z)}{\partial(u, v, w)} \times \frac{\partial(u, v, w)}{\partial(x, y, z)} = 1$$

$$= 4 \sec^2 v \times \frac{1}{4 \sec^2 v}$$

$$\Rightarrow \frac{\partial(x, y, z)}{\partial(u, v, w)} \times \frac{\partial(u, v, w)}{\partial(x, y, z)} = 1$$

4. (c) The time 'T' of a complete oscillation of a simple pendulum of length 'L' is governed by

the equation $T = 2\pi \sqrt{\frac{L}{g}}$, g is constant. Find the

approximate error in the calculated value of T corresponding to an error of 2% in the value of L.

Ans. Here $T = 2\pi \sqrt{\frac{L}{g}}$

$$\therefore \% \text{ error in } L = 2\%$$

$$\Rightarrow \frac{8L}{L} \times 100 = 2$$

$$\Rightarrow \frac{8L}{L} = \frac{2}{100}$$

$\therefore g$ is constant

$$\Rightarrow \frac{sg}{g} = 0.$$

$$\text{Now, } T = 2\pi \sqrt{\frac{L}{g}}$$

$$\Rightarrow \log T = \log 2\pi + \frac{1}{2}(\log L - \log g)$$

$$\Rightarrow \frac{sT}{T} = 0 + \frac{1}{2} \left(\frac{sL}{L} - \frac{sg}{g} \right)$$

$$\Rightarrow \frac{sT}{T} = \frac{1}{2} \times \frac{2}{100}$$

$$\Rightarrow \frac{sT}{T} = \frac{1}{100}$$

\therefore % error in $T = 1\%$.

Q. 5. (a) Determine 'b' such that the system of homogeneous equation $2x + y + 2z = 0$; $x + y + 3z = 0$; $4x + 3y + bz = 0$ has (i) Trivial solution (ii) Non-trivial solution. Find the Non-trivial solution using matrix method.

Ans. Given system of equation is

$$2x + y + 2z = 0$$

$$x + y + 3z = 0$$

$$4x + 3y + bz = 0$$

$$\text{Coefficient matrix } A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & b \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & b \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & b \end{bmatrix} R_3 \rightarrow R_3 - R_2$$

(i) For trivial solution,

Rank $A =$ No. of unknowns $= 2$

It is possible if $b - 8 \neq 0$

$$\Rightarrow b \neq 8$$

i.e. For trivial solution $b \neq 8$.

(ii) For non-trivial soln.

Rank $A <$ No. of unknowns $= 2$

It is possible if $b - 8 = 0$

$$\Rightarrow b = 8$$

i.e., for non trivial soln., $b = 8$.

To obtain this soln, put $b = 8$ in A , we get

$$A \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Rewriting the system,

$$x + y + 3z = 0$$

$$-y - 4z = 0$$

$$\text{Let } z = k$$

$$\Rightarrow y = -4k$$

$$\therefore x + y + 3z = 0$$

$$\Rightarrow x - 4k + 3k = 0$$

$$\Rightarrow x = k$$

$$\therefore \text{Desired non-trivial soln. is } \begin{cases} x = k \\ y = -4k \\ z = k \end{cases}$$

Q. 5. (b) Verify Cayley-Hamilton theorem for

the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ and hence find A^{-1} .

Ans. Given matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Characteristic eqⁿ of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-1-\lambda) - 4 = 0$$

$$\Rightarrow -1 - \lambda + \lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 5 = 0$$

To verify Cayley Hamilton theorem, we have to show

$$A^2 - 5I = 0$$

Now,

$$A^2 = AA$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & 2-2 \\ 2-5 & 4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Now,

$$A^2 - I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0$$

$$\Rightarrow A^2 - 5I = 0$$

Thus Cayley Hamilton theorem is verified.

To find A^{-1} , we multiply above relation by A^{-1} .

$$\text{We get } A^{-1}(A^2 - 5I) = 0A^{-1}$$

$$\Rightarrow A - 5A^{-1} = 0 \quad [\because IA^{-1} = A^{-1}]$$

$$\Rightarrow A^{-1} = \frac{1}{5}A$$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Q. 5. (c) Find the eigen value and corresponding eigen vectors of the matrix

$$I = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Ans. Given matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Characteristic eqn. of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-5-\lambda)(-2-\lambda) - 4 = 0$$

$$\Rightarrow (5+\lambda)(2+\lambda) - 4 = 0$$

$$\Rightarrow 10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow \lambda^2 + 6\lambda + \lambda + 6 = 0$$

$$\Rightarrow \lambda(\lambda+6) + 1(\lambda+6) = 0$$

$$\Rightarrow (\lambda+1)(\lambda+6) = 0$$

$$\Rightarrow \lambda = -1, -6.$$

which are eigen values.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be given vector of A corresponding

to eigen value λ .

$$\Rightarrow (A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = -1$, (1) gives

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -4x_1 + 2x_2 = 0$$

$$\text{or } 2x_1 - x_2 = 0$$

$$\& 2x_1 - x_2 = 0$$

$$\text{Let } x_1 = k, \Rightarrow x_2 = 2k$$

$$\therefore \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 2k \end{bmatrix}$$

$$k = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = -6$, (1) gives

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 1x_1 + 2x_2 = 0$$

$$\text{and } 2x_1 + 4x_2 = 0$$

$$\text{or } x_1 + 2x_2 = 0$$

$$\text{Let } x_2 = k_1 \Rightarrow x_1 = -2k_1$$

$$\therefore \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2k \\ k \end{bmatrix}$$

$$k = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Thus eigen value of given matrix are -1 and -6 .

Corresponding eigen vectors are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ respectively.

Q. 6. (a) Find the directional derivative of $\nabla(\nabla f)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $f = 2x^3y^2z^4$.

Ans. To find directional dervatve of $\nabla - \nabla f$

where $f = 2x^3y^2z^4$.

$$\nabla \cdot \nabla f = \nabla^2 f$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Now,

$$\frac{\partial f}{\partial x} = 6x^2y^2z^4$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = 12xy^2z^4$$

$$\text{Also } \frac{\partial f}{\partial y} = 4x^3yz^4$$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = 4x^3z^4$$

$$\text{and } \frac{\partial f}{\partial z} = 24x^3y^2z^2$$

$$\Rightarrow \nabla \cdot \nabla f = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

$$\text{Let } \phi = \nabla \cdot \nabla f$$

$$\Rightarrow \phi = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

\therefore Given surface is

$$xy^2z = 3x + z^2$$

$$\Rightarrow xy^2z - 3x - z^2 = 0$$

We shall find normal \vec{n} to this surface.

$$\text{Let } g = xy^2z - 3x - z^2$$

$$\Rightarrow \nabla g = \hat{i} \frac{\partial g}{\partial x} + \hat{j} \frac{\partial g}{\partial y} + \hat{k} \frac{\partial g}{\partial z}$$

$$\nabla g = \hat{i}[y^2z - 3] + \hat{j}[2xyz] + \hat{k}[xy^2 - 2z]$$

at given point $(1, -2, 1)$

$$(\nabla g)_{(1, -2, 1)} = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$\Rightarrow \quad \vec{n} = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$\Rightarrow \quad |\vec{n}| = \sqrt{(1)^2 + (-4)^2 + (2)^2}$$

$$\Rightarrow \quad |\vec{n}| = \sqrt{21}$$

$$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|}$$

$$= \frac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$$

To calculate directional derivative of $\nabla \cdot \nabla f = \phi$, we shall first calculate $\nabla \phi$.

Now,

$$\begin{aligned} \nabla \phi &= \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \\ &= \hat{i}[12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2] \\ &\quad + \hat{j}[24xyz^4 + 48x^3yz^2] \\ &\quad + \hat{k}[48xy^2z^3 + 16x^3z^3 + 48x^3y^2z] \end{aligned}$$

At (1, -2, 1)

$$\begin{aligned} (\nabla \phi)_{(1,-2,1)} &= \hat{i}(48 + 12 + 28.8) + \hat{j}(-48 - 96) \\ &\quad + \hat{k}(192 + \quad \quad 92) \end{aligned}$$

$$(\nabla \phi)_{(1,-2,1)} = 48\hat{i} - 144\hat{j} + 400\hat{k}$$

Desired directional derivative = $\nabla \phi \cdot \hat{n}$

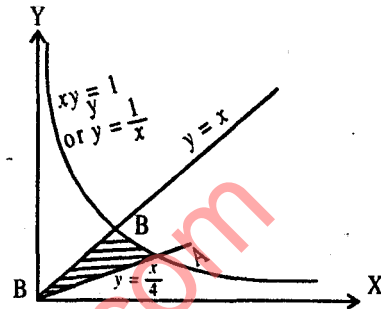
$$\begin{aligned} &= (348\hat{i} + 144\hat{j} + 400\hat{k}) \cdot \frac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k}) \\ &= \frac{1}{\sqrt{21}}(348 + 576 + 800) = \frac{1724}{\sqrt{21}} \end{aligned}$$

Q. 6. (b) Using Green's theorem, find the area of the region in the first quadrant bounded by the curves

$$y = x, y = \frac{1}{x}, y = \frac{x}{4}$$

Ans. Given curves are $y = x, y = \frac{1}{x}, u = \frac{x}{4}$.

Region bounded by these curves in first quadrant is shown in the following figure.



Required area = $\frac{1}{2} \int_C (xdy - ydx)$, where C represents the boundary of area.

By Green's theorem,

\Rightarrow Required area

$$\begin{aligned} &= \frac{1}{2} \int_{OA} (xdy - ydx) + \frac{1}{2} \int_{AB} (xdy - ydx) \\ &\quad + \frac{1}{2} \int_{BO} (xdy - ydx) \dots (1) \end{aligned}$$

Now,

Along OA, $y = \frac{x}{4}$

$$\Rightarrow \quad dy = \frac{dx}{4}$$

$$\therefore \int_{OA} (xdy - ydx) = \int_{OA} \left(\frac{xdx}{4} - \frac{xdx}{4} \right)$$

$$\Rightarrow \int_{OA} (xdy - ydx) = 0$$

Along AB, $y = \frac{1}{x}$

$$\Rightarrow dy = -\frac{1}{x^2} dx$$

$$\Rightarrow \int_{AB} (xdy - ydx) = \int_{AB} x \left(\frac{-1}{x^2} \right) dx - \frac{1}{x} dx$$

$$= -2 \int_{AB} \frac{1}{x} dx$$

To evaluate this integral, we shall first calculate value of x at A and B.

Now,

At A, $y = \frac{x}{4} \& y = \frac{1}{x}$

$$\Rightarrow \frac{x}{4} = \frac{1}{x}$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = 2$$

At B, $y = x \& y = \frac{1}{x}$

$$\Rightarrow x = \frac{1}{x}$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x = 1$$

$$\Rightarrow \int_{AB} (xdy - ydx) = -2 \int_2^1 \frac{1}{x} dx$$

$$= -2 [\log x]_2^1$$

$$= 2 \log 2 \quad [\because \log 1 = 0]$$

Also,

Along BO, $y = x$

$$\Rightarrow dy = dx$$

$$\Rightarrow \int_{BO} (xdy - ydx) = \int_{BO} xdx - xdx$$

$$= 0$$

Thus, putting above values in (1),

Required area $= \frac{1}{2}(0) + \frac{1}{2}(2 \log 2) + \frac{1}{2}(0)$

$$= \log 2$$

6.(c) Prove that $(y^2 - z^2 + 3yz)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational.

Ans. Given vector

$$\vec{V} = (y^2 - z^2 + 3yz)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$$

Here,

$$V_1 = y^2 - z^2 + 3yz$$

$$V_2 = 3xz + 2xy$$

$$V_3 = 3xy - 2xz + 2z$$

For \vec{V} to be irrotational, $\nabla \times \vec{V} = 0$.

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$$

Now,

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \hat{i}[3x - 3x] - \hat{j}[3y - 2z + 2z - 3y] + \hat{k}[3z + 2y - 2y - 3z]$$

$$= 0$$

$\Rightarrow \vec{V}$ is irrotational.

For \vec{V} to be solenoidal $\nabla \cdot \vec{V} = 0$

Now,

$$\nabla \cdot \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$$= 0 + 2x - 2x + 2$$

$$= 2$$

$$\therefore \nabla \cdot \vec{V} \neq 0$$

$\Rightarrow \vec{V}$ is not solenoidal

Hence given question is wrong.

7. (a) Changing the order of integration of

$$\int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx \, dy$$

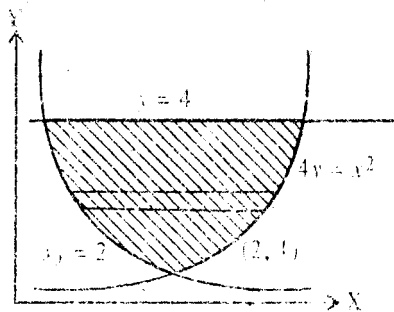
Show that $\int_0^\infty \left(\frac{\sin nx}{x} \right) dx = \frac{\pi}{2}$

Ans. See Q.5(a) of 2003-04.

7. (b) Determine the area bounded by the curves

$xy = 2$, $4y = x^2$ and $y = 4$.

Ans. Region bounded by given curves is shown in following figure.



Required area = $\iint_R dx \, dy$

where R is the shaded region.

Here limits of x are $\frac{2}{y}$ to $\sqrt{4y}$ and for y are

2 to 4.

$$\begin{aligned} \Rightarrow \text{Required area} &= \int_{y=2}^4 \int_{x=\frac{2}{y}}^{\sqrt{4y}} dx \, dy \\ &= \int_{y=2}^4 \left(2\sqrt{y} - \frac{2}{y} \right) dy \\ &= 2 \left[\frac{2}{3} y^{3/2} - \log y \right]_{y=2}^4 \end{aligned}$$

$$\begin{aligned} &= 2 \left[\frac{2}{3} (4)^{3/2} - \log 4 \right] \\ &\quad - 2 \left[\frac{2}{3} (4)^{3/2} - \log 2 \right] \\ &= 2 \left[\frac{2}{3} (2^2)^{3/2} - \log 2^2 \right] - 2 \left(\frac{2}{3} \right) \\ &= 2 \left[\frac{2}{3} \cdot 2^3 - 2 \log 2 \right] - \frac{4}{3} \\ &= \frac{32}{3} - 2 \log 2^2 - \frac{4}{3} \\ &= \frac{28}{3} - 4 \log 2 \end{aligned}$$

Q. 7. (c) For a β function, show that $\beta(p, q) = \beta(p+1, q) + \beta(p, q+1)$

Ans. R.H.S = $\beta(p+1, q) + \beta(p, q+1)$

$$\begin{aligned} &= \frac{\overline{p+1} \overline{q}}{\overline{p+q+1}} + \frac{\overline{p} \overline{q+1}}{\overline{p+q+1}} \\ &\quad \left[\because \beta(m, n) = \frac{\overline{m} \overline{n}}{\overline{m+n}} \right] \\ &= \frac{\overline{p} \overline{q} \overline{q}}{\overline{p+q+1}} + \frac{\overline{p} \overline{q} \overline{q}}{\overline{p+q+1}} \\ &\quad \left[\because \overline{n+1} = \overline{n} \overline{n} \right] \\ &= (p+q) \frac{\overline{p} \overline{q}}{\overline{p+q+1}} \\ &= \frac{(p+q) \overline{p} \overline{q}}{(p+q) \overline{p+q}} \quad \left[\because \overline{n+1} = \overline{n} \overline{n} \right] \\ &= \frac{\overline{p} \overline{q}}{\overline{p+q}} \\ &= \beta(p, q) = \text{L.H.S.} \end{aligned}$$

Hence proved